

# Calculation of density fluctuation in inflationary epoch

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Starting from an initial state of thermal equilibrium, we derive an expression for the quantum fluctuation in the energy density during the inflationary epoch in terms of the mode functions for the inflaton field. The effect of this particular initial state is not washed out in the final formula, contrary to what is usually believed. Numerically, however, the effect is completely negligible, validating the use of the two point function in the vacuum state. We also point out the requirement of conventional quantum field theory during inflation, that the quantum fluctuation in a wavelength must be evaluated, at the latest, when the wavelength crosses the Hubble length, in contrast to the usual practice in the literature.

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## I. INTRODUCTION

The most attractive aspect of inflationary models of the early universe [1] is their potential to predict the present day density inhomogeneity from first principles [2]. In these models it is possible to calculate quantum fluctuation in the energy density on the homogeneous background in a region within the causal horizon (given by the Hubble length) during the inflationary epoch. This fluctuation provides the initial spectrum of density perturbation. As the region inflates into the observed universe or bigger, its propagation through different eras can be followed till the present time by the equations of linear perturbation theory of classical gravity [3].

In this work we consider some points in the calculation of quantum fluctuation during inflation. The basic ingredient is the expectation value of the product of two scalar field operators at a time when considerable inflation has already taken place. It is generally believed that as the inflation proceeds, the effects of all scales associated with a particular initial state tend to be wiped out, retaining only the extremely high energies associated with quantum fluctuations in the vacuum. So the expectation value is evaluated for the homogeneous background, which corresponds to the vacuum state of quantum field theory.

Here we investigate how far the density fluctuation is actually independent of the initial condition prevailing at the beginning of inflation. For this purpose, we start with a definite initial state, namely that of thermal equilibrium. The first attempt in this direction was by Guth and Pi [4]. We discuss it here in a more general framework [5], [6]. The thermal propagator can be followed till the time when the quantum fluctuations are evaluated.

The existence of an initial thermal equilibrium distribution of particles, at least for the high wave numbers needed for the calculation of density fluctuation, appears quite probable. Even if the collision rates among the particles are too small to produce such a state, there could be another mechanism at work. As pointed out by Weinberg [7], the strong gravitational interaction at very early times would bring about thermal equilibrium, which, as we shall show, could be maintained at least till the beginning of inflation.

The other point we discuss is the time at which quantum fluctuation must be evaluated. It relates to the applicability of quantum field theory in curved space-time. As emphasised by DeWitt [8], conventional quantum field theory requires the mode functions to be oscillatory in time, allowing positive and negative frequencies to be identified. While on flat space-time such modes naturally arise for field theories describing physical particles, their existence is not guaranteed on space-times with non-zero curvature. The reason is that the curvature gives rise to a damping-like term in the equation of motion for the mode functions. In the inflationary period this makes a mode oscillatory or damped, according as the associated physical wavelength is smaller or bigger than the Hubble length. During this period physical wavelengths grow at a tremendous rate, while the Hubble length remains constant or approximately so. Thus even a wavelength lying initially deep inside the Hubble length would eventually go outside this length. So this time of exit marks the latest time at which we can evaluate the quantum fluctuation belonging to that particular wavelength.

In the literature, however, quantum fluctuations are actually evaluated at a time, when the modes have evolved well outside the the Hubble length, so as to be frozen [9]. Of course, the complete problem of predicting the density fluctuation at the time of Hubble length reentry in a later radiation or matter dominated phase does involve the

evolution of the fluctuation over a much longer period of time. But the question at hand is where the quantum fluctuation can be evaluated reliably.

In Sect. II we review the derivation of the thermal scalar propagator in the early universe. In particular, we show the behaviour of mode functions as the wavelengths grow and discuss the validity of the assumed initial thermal equilibrium state. In Sect. III we write the formula for density fluctuation in terms of the mode functions and discuss its dependence on the initial condition. In Sect. IV we consider the simple, original model of extended inflation as an example [10]. Here we review the homogeneous classical solution for the inflaton field and set up quantum theory in this background. Finally, our concluding remarks are contained in Sect. V.

## II. FINITE TEMPERATURE SCALAR PROPAGATOR

Consider a region of space in the early universe well within the causal horizon. It can then be taken to be homogeneous and isotropic, admitting the (spatially flat) line element,

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2, \quad (2.1)$$

where the scale factor  $a(t)$  describes the expansion of the region of the universe. It constitutes the background space-time, which is perturbed by quantum fluctuations. The action for the scalar field in this space-time may be generally written as,

$$S_\phi = \frac{1}{2} \int d^3x dt a^3(t) \left\{ \dot{\phi}^2 - \frac{1}{a^2(t)} (\nabla \phi)^2 - \mu^2(t) \phi^2 - \lambda_1(t) \phi^3 - \lambda_2(t) \phi^4 + \dots \right\}. \quad (2.2)$$

Here we have already shifted the scalar field by the classical, homogeneous field. The dots indicate interaction terms, if any, of the scalar field with other (gauge and matter) fields.

We assume the different species of particles to be in thermal equilibrium around some initial time  $t_0$ , which we conveniently take to be the time of transition of the radiation dominated phase to the inflationary phase. In particular, the scalar particles belonging to  $\phi(x)$  are also assumed to be in thermal equilibrium. (This assumption will be examined at the end of this section.) The density matrix is then given by

$$\rho = e^{-\beta_0 \mathcal{H}(t_0)} / \text{Tr} e^{-\beta_0 \mathcal{H}(t_0)}, \quad (2.3)$$

where  $1/\beta_0 = T(t_0)$  is the temperature at time  $t_0$ . The explicit time dependence of the Hamiltonian  $\mathcal{H}(t)$  arises from that of the scale factor and the homogeneous classical field. Note that the density matrix is constant in the Heisenberg representation. Thus once the system is in the thermal equilibrium state, the thermal propagator continues to hold even when the system deviates from this state.

To describe the time evolution of the system, it is most convenient to use the real time formulation of thermal field theory [11]. In the context of the early universe, the time path  $C$  in the action integral consists of three segments as shown in Fig.1 [5]. The points on it may be labelled by a complex parameter  $\tau$  such that

$$\tau = \begin{cases} t, & \text{on } C_1 \text{ and } C_2 \\ t_0 - it, & \text{on } C_3. \end{cases} \quad (2.4)$$

The action in the path integral corresponding to the segments  $C_1$  and  $C_2$  is in Minkowski space, while it is Euclidean on  $C_3$ . It should be noted, however, that the scale factor is not continued to Euclidean space on  $C_3$  : since the Hamiltonian for the segment is  $\mathcal{H}(t_0)$ , the scale factor remains fixed at  $t_0$ .

We now review the derivation of the thermal propagator [5], [6]. After a partial integration, the quadratic terms in  $S_\phi$  becomes [12],

$$S_0 = -\frac{1}{2} \int_C d\tau \int d^3x \phi D \phi + \text{boundary terms}, \quad (2.5)$$

where

$$D = \begin{cases} a^3 \left( \frac{d^2}{d\tau^2} + 3 \frac{\dot{a}}{a} \frac{d}{d\tau} + \omega^2 \right), & \omega^2(\tau) = -\frac{1}{a^2(\tau)} \nabla^2 + \mu^2(\tau), \quad \tau \in C_1, C_2 \\ a_0^3 \left( \frac{d^2}{d\tau^2} + \omega_0^2 \right), & \omega_0^2 = -\frac{1}{a_0^2} \nabla^2 + \mu^2(t_0), \quad \tau \in C_3. \end{cases} \quad (2.6)$$

We use the abbreviations,  $a_0 = a(t_0)$ ,  $\omega_0 = \omega(t_0)$ ,  $T_0 = T(t_0)$ . For the boundary terms to vanish, it is necessary that both  $\phi$  and  $\frac{d\phi}{d\tau}$  match at the joining of the segments  $C_1$  and  $C_2$ , of  $C_2$  and  $C_3$  as well as at the free ends of  $C_3$  and  $C_1$ .

The (time ordered) thermal propagator  $\text{Tr} \rho T \phi(x) \phi(x')$  will be denoted by  $G_\beta(x, \tau; x', \tau')$  or  $< T \phi(x) \phi(x') >$ . It satisfies

$$DG_\beta(\vec{x}, \tau; \vec{x}', \tau') = -i\delta^3(\vec{x} - \vec{x}')\delta(\tau - \tau') , \quad (2.7)$$

with boundary conditions following from the matching of  $\phi$  and  $\frac{d\phi}{d\tau}$  mentioned above. For the spatial Fourier transform of the propagator, defined by

$$G_\beta(\vec{x}, \tau, \tau') = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} G_\beta(\vec{k}, \tau, \tau') , \quad (2.8)$$

it reduces to

$$DG_\beta(\vec{k}; \tau, \tau') = -i\delta(\tau - \tau') , \quad (2.9)$$

where  $-\nabla^2$  appearing in the expressions for  $\omega^2(\tau)$  and  $\omega_0^2$  is to be replaced now by  $k^2$ .

To construct the propagator we first find the mode functions. On the contour  $C_3$  they satisfy

$$\left( \frac{d^2}{d\tau^2} + \omega_0^2 \right) h^\pm(\tau) = 0 , \quad (2.10)$$

giving

$$h^\pm(\tau) = \frac{1}{\sqrt{2\omega_0 a_0^3}} e^{\mp i\omega_0 \tau} , \quad \tau = t_0 - it \in C_3 . \quad (2.11)$$

The normalization satisfies the Wronskian condition,  $\dot{h}^+(\tau)h^-(\tau) - \dot{h}^-(\tau)h^+(\tau) = -i/a_0^3$ . The mode functions on the real segments  $C_1$  and  $C_2$  are the solutions of

$$\left( \frac{d^2}{d\tau^2} + 3\frac{\dot{a}}{a}\frac{d}{d\tau} + \omega^2(\tau) \right) g^\pm(\tau) = 0 , \quad \tau = t \in C_{1,2} , \quad (2.12)$$

with normalisation fixed again by the Wronskian condition,  $\dot{g}^+(\tau)g^-(\tau) - \dot{g}^-(\tau)g^+(\tau) = -i/a^3(\tau)$ . To see the nature of these solutions, we put

$$g^\pm(\tau) = a^{-3/2} \bar{g}^\pm(\tau) , \quad (2.13)$$

where  $\bar{g}$  satisfies

$$\left( \frac{d^2}{d\tau^2} + \bar{\omega}^2(\tau) \right) \bar{g}^\pm(\tau) = 0 , \quad (2.14)$$

with

$$\bar{\omega}^2(t) = \frac{k^2}{a^2} + \mu^2 - \frac{9}{4} \left( H^2 + \frac{2}{3} \dot{H} \right) , \quad H(t) = \frac{\dot{a}(t)}{a(t)} . \quad (2.15)$$

For a power law behaviour of the scale factor,  $a(t) \sim t^p$ , it becomes

$$\bar{\omega}^2(t) = \frac{k^2}{a^2} + \mu^2 - \frac{9}{4} \left( 1 - \frac{2}{3p} \right) H^2 . \quad (2.16)$$

It is now simple to identify the modes, which belong to conventional quantum field theory. The magnitude of  $\mu(t)$  is usually small compared to  $H(t)$ . Thus in the radiation dominated phase ( $p = \frac{1}{2}$ ),  $\bar{\omega}^2$  is positive for all values of  $k$  and it may be possible to define oscillatory modes belonging to positive and negative frequencies, at least in a quasi-static way. We show below that this is indeed the case around the time  $t_0$ , when we can solve (2.14) in the JWKB approximation to get [13],

$$\bar{g}^\pm(\tau) = \frac{1}{\sqrt{2\bar{\omega}(\tau)}} e^{\mp i \int_{t_0}^\tau d\tau' \bar{\omega}(\tau')} , \quad \tau \simeq t_0 . \quad (2.17)$$

We thus have a valid quantum field theory around the time  $t_0$ .

But in the inflationary phase ( $p \gg 1$ ), a mode is oscillatory only if its physical wavelength  $2\pi a(t)/k$  is small compared to the Hubble length  $H^{-1}(t)$ . As inflation progresses, the scale factor increases enormously, while  $H(t)$  is approximately constant. Thus more and more modes go out of Hubble length and behave as damped waves, having no interpretation in quantum field theory.

The solutions  $g^\pm(\tau)$  and  $h^\pm(\tau)$  may now be joined to form the functions  $f^\pm(\tau)$  on the entire contour  $C$ ,

$$f^\pm(\tau) = \begin{cases} g^\pm(\tau), & \tau \in C_{1,2} \\ h^\pm(\tau), & \tau \in C_3 \end{cases} \quad (2.18)$$

By definition,  $f^\pm(\tau)$  obey the continuity conditions relating the segments  $C_1$  and  $C_2$ . Using Eqs (2.11) and (2.17), we see that the conditions connecting  $C_2$  and  $C_3$  are also well satisfied if  $H(t_0)$  is small compared to  $k/a(t_0)$  [14]. A particular solution to (2.9) may now be written as

$$G_0(k; \tau, \tau') = f^+(\tau)f^-(\tau')\theta_c(\tau - \tau') + f^+(\tau')f^-(\tau)\theta_c(\tau' - \tau), \quad (2.19)$$

where  $\theta_c$  is a step function on the contour. It satisfies the continuity conditions at the junctions of segments  $C_1$  and  $C_2$  as well as of  $C_2$  and  $C_3$ , because  $f^\pm(\tau)$  does it. To satisfy the remaining (thermal) continuity condition at the ends of  $C_1$  and  $C_3$ , we add to it the most general solution of the homogeneous equation,

$$G_\beta(k; \tau, \tau') = G_0(k, \tau, \tau') + \sum_{i,j=1}^2 f^i(\tau)\Lambda^{ij}f^j(\tau'). \quad (2.20)$$

The superscript  $(\pm)$  on the mode functions are replaced temporarily by 1 and 2 to use matrix notation. The  $2 \times 2$  constant coefficient matrix  $\Lambda$  is uniquely determined by the thermal conditions [15]. We get

$$G_\beta(k, \tau, \tau') = f^+(\tau)f^-(\tau')\{\theta_c(\tau - \tau') + n(\omega_0)\} + f^-(\tau)f^+(\tau')\{\theta_c(\tau' - \tau) + n(\omega_0)\}, \quad (2.21)$$

where  $n(\omega_0)$  is the bosonic distribution function

$$n(\omega_0) = (e^{\beta_0\omega_0} - 1)^{-1}. \quad (2.22)$$

For tree level calculations we need the Green's function only on the real axis  $C_1$ . Writing henceforth  $\tau = t$ , this is given by

$$\langle \phi(\vec{x}, t)\phi(\vec{x}', t') \rangle = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} (1 + n(\omega_0)) g^+(t) g^-(t'), \quad t > t', \quad (2.23)$$

where the mode functions  $g^\pm(t)$  are solutions of (2.12).

We now come back to the assumption of the initial thermal equilibrium state. Such an initial state can be ensured in an expanding universe if collisions among particles occur at a rate faster than the expansion rate of the universe. While this condition holds for species interacting through (relatively large) gauge coupling, it may not hold for particles of the inflaton field, which is a gauge singlet and has weak self-interaction. We discuss below the other mechanism, mentioned in the introduction, which could give rise to thermal equilibrium around the time  $t_0$ .

At the Planck time  $t_P$ , the strong gravitational interaction brings all species into thermal equilibrium [7]. If the system is quantised at this time in a cubic volume with sides small compared to the Hubble length, the longest wavelength will be well inside this length, *i.e.*

$$\frac{k}{a(t_P)} > \pi H(t_P), \quad (2.24)$$

even for the smallest wavenumber. Then eq (2.16) simplifies to

$$\bar{\omega}(t_P) = \frac{k}{a(t_P)}, \quad (2.25)$$

to a good approximation and the density distribution becomes

$$n(\omega(t_P)) = \frac{1}{e^{k/a(t_P)T(t_P)} - 1}. \quad (2.26)$$

We now bring the inequality (2.24) to the time  $t_0$ ,

$$\frac{k}{a(t_0)} > \frac{a(t_P)}{a(t_0)} \frac{H(t_P)}{H(t_0)} \pi H(t_0) = \frac{m_P}{T_0} \pi H(t_0), \quad (2.27)$$

where  $m_P$  is the Planck mass. In the last step we have used the radiation dominated solution for  $a(t)$ . (See eqns. (4.6-7) below.) The temperature  $T_0$  is given by the grand unification scale,  $T_0 \sim 10^{15} \text{ GeV}$ , so that  $m_P/T_0 \sim 10^5$ . Thus the relation (2.25) at time  $t_P$  continues to hold throughout the radiation dominated phase; in fact, it becomes more and more accurate as  $t$  increases from  $t_P$ . Clearly the equilibrium distribution (2.26) established at time  $t_P$  is well maintained at least till the time  $t_0$ .

### III. DENSITY FLUCTUATION FORMULA

The density inhomogeneity (at time  $t$ ) is measured by the mean square fluctuation in the density function  $\rho(\vec{x}, t)$  [16] [17],

$$\left(\frac{\delta\rho}{\rho}\right)^2 = \left\langle \left(\frac{\rho(\vec{x}, t) - \bar{\rho}(t)}{\bar{\rho}(t)}\right)^2 \right\rangle_x, \quad (3.1)$$

where  $\langle \dots \rangle_x$  denotes averaging over space and  $\bar{\rho}$  is the homogeneous background density,  $\bar{\rho} = \langle \rho(\vec{x}, t) \rangle_x$ . In the inflationary scenario, this fluctuation in the early universe is supposed to arise from quantum fluctuation. We may calculate the latter by evaluating an expression similar to (3.1), with  $\rho(\vec{x}, t)$  replaced by the corresponding operator  $\hat{\rho}(\vec{x}, t)$  and the averaging process by taking the expectation value in an appropriate state. There is, however, a technical problem with this quantum version, as it involves the product of  $\hat{\rho}(\vec{x}, t)$  with itself at the same space-time point, which is not defined in quantum field theory. The problem may be avoided by taking the smeared density function [4],

$$\rho_l(\vec{x}, t) = N \int d^3y e^{-y^2/2l^2} \rho(\vec{x} + \vec{y}, t), \quad (3.2)$$

where  $l$  is an arbitrary smearing length and  $N$  an irrelevant normalization factor. The fluctuation in  $\rho_l$  is given by

$$\left(\frac{\delta\rho_l}{\rho_l}\right)_c^2 = \left\langle \left(\frac{\rho_l(\vec{x}, t) - \bar{\rho}_l(t)}{\bar{\rho}_l(t)}\right)^2 \right\rangle_x, \quad (3.3)$$

where the subscript  $c$  stands for classical. The corresponding quantum fluctuation, denoted by the subscript  $q$ , is now well defined,

$$\left(\frac{\delta\rho_l}{\rho_l}\right)_q^2 = \left\langle \left(\frac{\hat{\rho}_l(\vec{x}, t) - \bar{\rho}(t)}{\bar{\rho}(t)}\right)^2 \right\rangle, \quad (3.4)$$

where  $\langle \dots \rangle$  stands for the expectation value in the initial thermal state defined by eqn (2.3).

To treat perturbation on different length scales, one writes

$$\rho(\vec{x}, t) = \bar{\rho}(t)(1 + \delta(\vec{x}, t)), \quad (3.5)$$

and Fourier analyzes the so-called density contrast,  $\delta(\vec{x}, t)$ ,

$$\delta(\vec{x}, t) = \frac{1}{\sqrt{V}} \sum_k \delta_k(t) e^{i\vec{k} \cdot \vec{x}}, \quad (3.6)$$

where  $V$  is a volume within the Hubble length to begin with. In the limit of large volume, (3.3) becomes

$$\left(\frac{\delta\rho_l}{\rho_l}\right)_c^2 = \int \frac{d^3k}{(2\pi)^3} |\delta_k(t)|^2 e^{-k^2 l^2}. \quad (3.7)$$

The energy density operator  $\hat{\rho}(\vec{x}, t)$  is the time-time component of energy momentum tensor,

$$T_{\mu\nu} = \partial_\mu \Psi \partial_\nu \Psi - g_{\mu\nu} \left( \frac{1}{2} g^{\mu\nu} \partial_\alpha \Psi \partial_\beta \Psi - V(\Psi) \right), \quad (3.8)$$

for the full scalar field  $\Psi(x)$ . Here the potential function  $V(\Psi)$  depends on the model considered. We shift  $\Psi(x)$  by the homogeneous classical field  $\psi(x)$ ,

$$\Psi(\vec{x}, t) = \psi(t) + \phi(\vec{x}, t), \quad (3.9)$$

such that for the quantum field  $\langle \phi(x) \rangle = 0$ . The  $\hat{\rho}(x)$  is given by

$$\hat{\rho}(x) = \bar{\rho}(t) + \hat{U}(x), \quad (3.10)$$

where

$$\bar{\rho}(t) = \frac{1}{2} \dot{\psi}^2 + V(\psi), \quad (3.11)$$

and

$$\hat{U}(x) = r(t)\phi(x) + s(t)\dot{\phi}(x), \quad (3.12)$$

to first order in  $\phi(x)$ . The coefficients  $r(t), s(t)$  in (3.12) depend on the classical field and other parameters in the potential  $V(\Psi)$ . Terms in  $\hat{U}$ , which are of higher order in  $\phi$  would give loop contribution to the density fluctuation and are neglected. Using (3.10), (3.12) and (2.23), the expectation value in (3.4) may be evaluated to give,

$$\begin{aligned} \left( \frac{\delta \rho_l}{\rho_l} \right)_q^2 &= \frac{1}{\bar{\rho}^2(t)} \int d^3x d^3y e^{-(x^2+y^2)/l^2} < \hat{U}(x, t) \hat{U}(y, t) > \\ &= \frac{1}{\bar{\rho}^2(t)} \int \frac{d^3k}{(2\pi)^3} (1 + n(\omega_0)) |r(t)g_k(t) + s(t)\dot{g}_k(t)|^2 e^{-k^2 l^2}. \end{aligned} \quad (3.13)$$

Comparing eqs (3.7) and (3.13) we get the desired result,

$$|\delta_k(t)|^2 = \frac{1}{\bar{\rho}^2} (1 + n(\omega_0)) |r(t)g_k(t) + s(t)\dot{g}_k(t)|^2. \quad (3.14)$$

Following our discussion in Sect. II, we evaluate it at time  $t_h$ , when the wavelength crosses the Hubble length,

$$\frac{k}{a(t_h)} = 2\pi H(t_h).$$

The treatment of the evolution of density perturbation outside the Hubble length is standard [16], [17]. At the time of its reentry within the Hubble radius of the post-inflationary radiation dominated epoch, it is given by [18],

$$\left( \frac{\delta \rho}{\rho} \right)_H = \frac{\sqrt{k^3(1 + n(\omega_0))} |rg_k(t_h) + s\dot{g}_k(t_h)|}{3\sqrt{2}\pi(\bar{\rho} + \bar{p})_{t_h}}, \quad (3.15)$$

where  $\bar{p}$  is the homogeneous pressure,  $\bar{p} = \frac{1}{2}\dot{\psi}^2 - V(\psi)$ .

It is simple to estimate  $n(\omega_0)$  in the range of  $k/a_0$ , which is of interest. We write

$$\frac{k}{a_0} = \frac{k}{a(t_p)} \cdot \frac{a(t_p)}{a(t_e)} \cdot \frac{a(t_e)}{a(t_0)}. \quad (3.16)$$

From the time  $t_e$  when the inflation ends till the present time  $t_p$ , the universe expands adiabatically, so that  $a(t_p)/a(t_e) \simeq T_0/T_p$ . The other ratio  $a(t_e)/a(t_0) \equiv Z$  gives the magnitude of inflation. We thus get

$$\frac{k}{a_0 T_0} = \frac{2\pi}{\lambda(t_p)} \frac{Z}{T_p} \sim \frac{1}{\lambda_{Mpc}} \cdot \frac{Z}{10^{25}}, \quad (3.17)$$

where  $T_p = 2.7K$  and  $\lambda_{Mpc}$  is  $\lambda(t_p)$  expressed in  $Mpc$ . The wavelengths of interest stretch over the range  $1 < \lambda_{Mpc} < 10^4$ . To solve the problems of the standard cosmology we need  $Z > 10^{25}$ . But actually  $Z$  exceeds this limit by many orders of magnitude in most of the models of inflation. Thus  $k/a_0 T_0$  is large in these models and we may set  $n(\omega_0) = 0$  in the expression (3.14).

We thus see that although the initial thermal equilibrium state does produce a factor in the expression for the density inhomogeneity, its magnitude turns out to be unity, justifying the use of zero temperature propagator for its evaluation. Nevertheless it is important to know the initial state, as there are other quantities, such as the duration of inflation, which may depend sensitively on it.

#### IV. EXAMPLE OF EXTENDED INFLATION

In this section we consider the well-studied, original model of extended inflation [10]. We first summarise, for the sake of completeness, the classical inflationary solution for the scalar field, which joins smoothly to the radiation dominated solutions before and after the inflation. We then write the Lagrangian for the quantum field in the background of this homogeneous field. The mode functions and the (vacuum) propagator has already been evaluated in a recent work [18]. Here we only discuss the possibility of establishing thermal equilibrium through collisions.

The model is based on the Brans- Dicke (BD) theory of gravity given by the action [19],

$$S = \int d^4x \sqrt{g} \left( \frac{R}{16\pi} \Phi + \frac{\omega}{16\pi} g^{\mu\nu} \frac{\partial_\mu \Phi \partial_\nu \Phi}{\Phi} + \mathcal{L}_{matter} \right). \quad (4.1)$$

The time dependence of the BD field  $\Phi$  makes the effective gravitational ‘constant’ vary with time.  $\omega$  is a dimensionless parameter.  $\mathcal{L}_{matter}$  represents the contribution of all other fields including the inflaton field  $\sigma$ ,

$$\mathcal{L}_{matter} = \frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - V(\sigma) + \dots . \quad (4.2)$$

This simple model is not realistic, however. The bubble nucleation problem can be solved for  $\omega \leq 25$ , while astrophysical observation constrains it to  $\omega > 500$ . Our purpose here is to examine in this toy model the possibility of realising thermal equilibrium as the initial state for the inflationary epoch. In the following we take  $\omega > 25$ , say.

The equations satisfied by the classical fields, *viz.* the scale factor  $a(t)$  and the homogeneous part  $\varphi(t)$  of the BD field  $\Phi(x)$  are [10],

$$\ddot{\varphi} + 3H\dot{\varphi} = \frac{8\pi}{2\omega + 3}(\rho - 3p) \quad , \quad H = \frac{\dot{a}}{a}, \quad (4.3)$$

$$H^2 = \frac{8\pi\rho}{3\varphi} + \frac{\omega}{6}\left(\frac{\dot{\varphi}}{\varphi}\right)^2 - H\left(\frac{\dot{\varphi}}{\varphi}\right) , \quad (4.4)$$

where the energy density  $\rho$  and the pressure  $p$  are generated by  $\mathcal{L}_{matter}$ . If the masses are small compared to the temperature, they are

$$\rho = \frac{\pi^2}{30}NT^4 + M^4, \quad p = \frac{\pi^2}{90}NT^4 - M^4, \quad (4.5)$$

where  $M^4 = V(0)$  is the false vacuum energy density and  $N$  counts the total number of effective degrees of freedom.

In the pre-inflationary radiation dominated epoch,  $\varphi(t)$  is constant [20],

$$\varphi(t) = \varphi(t_0) \quad , \quad a(t) \propto \sqrt{t}, \quad (4.6)$$

giving

$$T^2t = \sqrt{\frac{45\varphi(t_0)}{16\pi^3N}}, \quad (4.7)$$

while in the inflationary epoch the solutions may be written as [10]

$$\varphi(t) = \varphi(t_0)\{1 + B(t - t_0)\}^2, \quad (4.8)$$

with

$$B\sqrt{\varphi}(t_0) = \frac{M^2}{\omega q}, \quad q = \sqrt{\frac{(6\omega + 5)(2\omega + 3)}{32\pi\omega^2}},$$

and

$$a(t) = a(t_0)\{1 + B(t - t_0)\}^{\omega+1/2}. \quad (4.9)$$

The two sets of solutions join at  $t_0$ , when we have approximately

$$\frac{\pi^2}{30}NT_0^4 \simeq M^4, \quad (4.10)$$

giving  $T_0 \simeq M$  and

$$T_0^2t_0 = \sqrt{\frac{45\varphi(t_0)}{16\pi^3N}}. \quad (4.11)$$

Equating the values of  $H(t_0)$  for the two solutions, we get another relation

$$\frac{1}{2t_0} = (\omega + \frac{1}{2})B. \quad (4.12)$$

It is no new constraint, coinciding with (4.11) to leading order in  $\omega$ .

We now assume as usual that the inflationary period ends instantly at time  $t_e$  in radiation domination again at a temperature  $T \sim T_0$ . Then  $\varphi(t)$  ceases to vary appreciably, so that we have

$$\varphi(t) = \varphi(t_e) = m_P^2, \quad t \geq t_e, \quad (4.13)$$

where  $m_P \equiv G^{-1/2}$  is the present value of the Planck mass. Using (4.8) and (4.9) we can relate  $\varphi(t_o)$  to the amount of inflation  $Z$ ,

$$\varphi(t_o) = m_P^2 Z^{-4/(2\omega+1)}, \quad Z = \frac{a(t_e)}{a(t_o)}. \quad (4.14)$$

To set up a semi-classical quantum theory in the background of the above solutions, one faces the problem of dealing with the non-standard form of the action (4.1). This problem may be avoided by going over to a new (Einstein) frame from the present (Jordan) frame by a conformal transformation of the metric. Denoting quantities in the new frame by a bar, the required transformation is [21],

$$\bar{g}_{\mu\nu}(\vec{x}, t) = \Omega^{-2}(t) g_{\mu\nu}(\vec{x}, t), \quad (4.15)$$

where

$$\Omega^2(t) = \frac{m_P^2}{\Phi(t)}. \quad (4.16)$$

One also introduces a field  $\Psi$  to replace  $\Phi$ ,

$$\Psi = \chi \ln \left( \frac{\Phi}{m_P^2} \right), \quad \chi = \sqrt{\frac{2\omega+3}{16\pi}} m_P, \quad (4.17)$$

which brings the kinetic term for  $\Phi$  in the canonical form. Assuming  $\sigma = \text{constant}$  in the inflationary phase, the new action becomes,

$$\bar{S} = \int d^4x \sqrt{\bar{g}} \left\{ \frac{\bar{R}}{16\pi m_P^2} + \frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi + V(\Psi) \right\}, \quad (4.18)$$

where  $V(\Psi) = M^4 e^{-2\Psi/\chi}$ . In the Einstein frame  $\Psi$  plays the role of the inflaton field.

The new metric may be brought back to the Robertson-Walker form by a redefinition of the time coordinate [21],

$$d\bar{t}^2 - \bar{a}^2(\bar{t}) d\vec{x}^2 = \Omega^{-2}(t) \{ dt^2 - a^2(t) d\vec{x}^2 \}, \quad (4.19)$$

giving

$$d\bar{t} = \Omega^{-1}(t) dt, \quad \bar{a}(\bar{t}) = \Omega^{-1}(t) a(t). \quad (4.20)$$

Using (4.16) and (4.20), the inflationary solutions (4.8-9) in the Jordan frame may be recast in the Einstein frame. Integrating the first equation in (4.20) and choosing the constant of integration properly, we get [22],

$$1 + C(\bar{t} - \bar{t}_0) = \{1 + B(t - t_0)\}^2, \quad C = \frac{2m_P B}{\sqrt{\varphi(t_0)}}. \quad (4.21)$$

Then the new scale factor is obtained from the second equation in (4.20),

$$\bar{a}(\bar{t}) = \bar{a}(\bar{t}_0) \{1 + C(\bar{t} - \bar{t}_0)\}^{(2\omega+3)/4}, \quad \bar{a}(\bar{t}_0) = a(t_0) \frac{\sqrt{\varphi(t_0)}}{m_P}. \quad (4.22)$$

The classical homogeneous part of  $\Psi$  is obtained from (4.8) and (4.17),

$$\psi(\bar{t}) = \psi(\bar{t}_0) + \chi \ln \{1 + C(\bar{t} - \bar{t}_0)\}, \quad \psi(\bar{t}_0) = \chi \ln \left( \frac{\varphi(t_0)}{m_P^2} \right). \quad (4.23)$$

These solutions could, of course, have been found by solving the equations obtained by varying the action (4.18) in the Einstein frame.

The free part of the action (4.18) is now in the standard form for developing quantum field theory. We introduce the quantum field  $\phi(x)$  by decomposing  $\Psi$  as,

$$\Psi(\vec{x}, t) = \psi(t) + \phi(\vec{x}, t). \quad (4.24)$$

The exponential potential  $V(\Psi)$  as such has no place among renormalizable quantum field theories. But we can treat it as an effective potential and retain terms up to fourth power in  $\phi$ . The action for  $\phi$  then becomes



$$\bar{S} = \int d^3x d\bar{t} \bar{a}^3 \left\{ \frac{1}{2} \left( \frac{d\phi}{d\bar{t}} \right)^2 - \frac{1}{2\bar{a}^2} (\nabla\phi)^2 - \frac{1}{2} \mu^2 \phi^2 - \lambda_1 \phi^3 - \lambda_2 \phi^4 \right\}, \quad (4.25)$$

where

$$\begin{aligned} \mu^2(\bar{t}) &\simeq \frac{12}{\omega} \bar{H}^2(\bar{t}), \\ \lambda_1(\bar{t}) &\simeq -\frac{8\sqrt{2\pi}}{\omega^{3/2}} \frac{\bar{H}^2(\bar{t})}{m_P}, \\ \lambda_2(\bar{t}) &\simeq \frac{16\pi}{\omega^2} \frac{\bar{H}^2(\bar{t})}{m_P^2}. \end{aligned} \quad (4.26)$$

The magnitudes of  $\mu^2$ ,  $\lambda_1$  and  $\lambda_2$  depend on  $1/\varphi(t_0)$ , the effective gravitational constant in the pre-inflationary, radiation dominated phase. The relation (4.14) is not useful in determining  $\varphi(t_0)$  as  $Z$  itself depends on the details of the models of inflation. However a lower bound on  $\varphi(t_0)$  may be obtained from an earlier requirement that  $\mu(t_0) < T_0$ , giving

$$\varphi(t_0) > \sqrt{\frac{32\pi}{\omega}} \cdot M m_P.$$

From the corresponding bounds on  $\lambda_1$  and  $\lambda_2$ , it is simple to find that the collision rate,  $\Gamma_{coll} < \left(\frac{4\pi}{3\omega}\right)^2 \left(\frac{M}{m_P}\right)^4 M$ , which is hopelessly small compared to the expansion rate,  $\Gamma_{exp} = \bar{H}(\bar{t}_0)$ . Even though collisions are totally ineffective to maintain thermal equilibrium, we have shown in Sect. II that gravitational interaction at very early times would ensure equilibrium distribution of the  $\varphi$  particles around the time  $t_0$ .

There are several earlier calculations of density fluctuation in this model [22], [23]. All are based on the result for the vacuum propagator for the scalar field, which we justify in Sect. III. However earlier works generally evaluate the quantum fluctuation outside the Hubble length, which following our discussion in Sect. II, cannot be interpreted properly in quantum field theory. In a recent work [18], we evaluate this density fluctuation at the time of Hubble length exit using the formula (3.15). We find a result, which is an order of magnitude bigger than the earlier ones.

## V. CONCLUSION

In the present work we assume the inflationary epoch to begin in a state of thermal equilibrium and study its effect on the quantum fluctuation in the energy density calculated during this epoch. This initial state including the scalar particles appears quite likely even if their self-interaction is too feeble to ensure it. We show that the thermal equilibrium established at very early times through the-then strong gravitational interaction would be maintained till the beginning of inflation. By evaluating the scalar field propagator with thermal boundary conditions, we find a result for the density fluctuation, which differs from the one calculated with the vacuum propagator by the factor  $\{1 + (e^{k_0/a_0 T_0} - 1)^{-1}\}$ . Clearly the factor does not go to unity as time passes but is a constant depending on the physical wave number and the temperature referred to the initial time  $t_0$ .

It turns out, however, that for wave numbers of interest in the present universe, this factor is unity in models where the amount of inflation exceeds by many orders of magnitude the minimal amount required to solve the problems of standard cosmology. Thus numerically the calculation of fluctuation in the vacuum state is justified.

We also point out that the conventional quantum field theory applies on curved space-time as long as the modes oscillate. This requires that we evaluate the quantum fluctuation, at the latest, when the corresponding wavelength crosses the Hubble length. Previous works [9], however, evaluate it as a rule for wavelengths well outside this length, where the modes freeze. As we have shown in a recent work [18], this difference in the calculation leads to an increase in the result by a factor of five for the model of extended inflation.

Finally we note that as long as the modes are within the Hubble length, they retain a thermal equilibrium distribution. Thus although the initial state of thermal equilibrium comprising all modes is not maintained during inflation, the modes relevant for the calculation of density fluctuation are those still in an equilibrium distribution.

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- [2] S. W. Hawking, Phys. Lett. **115B**, 295 (1982),  
A. H. Guth and S. -Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982),  
J. M. Bardeen, P. J. Steinhardt and M. S. Turner, Phys. Rev. **D28**, 679 (1983).
- [3] J. M. Bardeen, Phys. Rev. **D32**, 1899 (1985).
- [4] A. H. Guth and S. -Y. Pi, Phys. Rev. **D32**, 1899 (1985).
- [5] G. Semenoff and N. Weiss, Phys. Rev. **D31**, 689 (1985), *ibid* 699 (1985).
- [6] H. Leutwyler and S. Mallik, Ann. Phys. **205**, 1 (1991);  
N. Banerjee and S. Mallik, Ann. phys. **205**, 29 (1991).
- [7] S. Weinberg, Phys. Rev. Lett. **42**, 850 (1979).
- [8] B. DeWitt, Phys. Rep. **19C**, 295 (1975).
- [9] J. E. Lidsey, A. R. Liddle, E. W. Kolb, J. Copeland, T. Barreiro and M. Abney, Rev. Mod. Phys. **69** 373 (1997).
- [10] C. Mathiazhagan and V. B. Johri, Class. Quant. Grav. **1**, 229 (1984),  
D. La and P. Steinhardt, Phys. Rev. Lett. **62**, 3761 (1989).
- [11] A. J. Niemi and G. W. Semenoff, Ann. Phys. **152** (1984), 105; Nucl. Phys. **230** [FS10] (1984), 181.  
See also, G. W. Semenoff and H. Umezawa, Nucl. Phys. **220** [FS 8] (1983), 186.
- [12] The quadratic action is generally improved by replacing  $\mu(t)$  with the effective mass the scalar particle acquires at finite temperature through the self- and gauge-interactions. But for the inflaton field, which has no gauge interaction and is only weakly self-coupled, this modification is not significant.
- [13] The approximation is valid for  $\dot{\omega} \ll \omega^2$ . In the present work where  $\mu(t)$  is assumed small compared to  $H(t)$ , this condition is equivalent to  $k/a(t) \gg H(t)$ . In ref. [6], it is ensured for all physical momenta in a different way, *viz*, by assuming a large thermal mass for the scalar particle.
- [14] Let us mention here the problem associated with the density matrix (2.3) defined sharply at time  $t_0$ . This procedure of instant thermalisation leads to additional short distance singularities in the propagator, not present at zero temperature, making the theory non-renormalisable. Here we thermalize the system in a static background with constant scale factor prior to  $t_0$  and then connect it smoothly to the actual scale factor. The resulting thermal propagator does not depend on the details of the interpolating scale factor, if the condition  $k/a(t_0) \gg H(t_0)$  is satisfied. See ref. [6] for more details.
- [15] It is interesting to note that although we insisted on oscillatory modes with positive and negative frequencies to define quantum field theory, the propagator (2.20) is invariant under a reparametrization mixing the mode functions,  $f^+(\tau) \rightarrow \alpha f^+(\tau) + \beta f^-(\tau)$ ,  $f^-(\tau) \rightarrow \gamma f^+(\tau) + \delta f^-(\tau)$ , the constant coefficients  $\alpha, \beta, \gamma$  and  $\delta$  satisfying  $\alpha\delta - \beta\gamma = 1$ . It is a property of the thermal propagator also on flat space-time.
- [16] E. W. Kolb and M. S. Turner, The Early Universe (Addison Wesley, 1989).
- [17] T. Padmanabhan, Structure formation in the universe ( Camb. Univ. Press, 1993).
- [18] S. Mallik and D. Rai Chaudhuri, Phys. Rev. **D56**, 625 (1997).
- [19] C. Brans and R.H. Dicke, Phys. Rev. **24**, 924 (1961); P. Jordan, Z. Phys. **157**, 112 (1959).
- [20] S. Weinberg, Gravitation and cosmology (John Wiley & Sons, 1972).
- [21] R. Holman, E. W. Kolb, S. L. Vadas, Y. Yang and E. J. Weinberg, Phys. Lett. **B237**, 37 (1990).
- [22] A. H. Guth and B. Jain, Phys. Rev. **D45**, 426 (1992).
- [23] E. W. Kolb, D. S. Salopek and M. S. Turner Phys. Rev. **D42**, 3925 (1990).

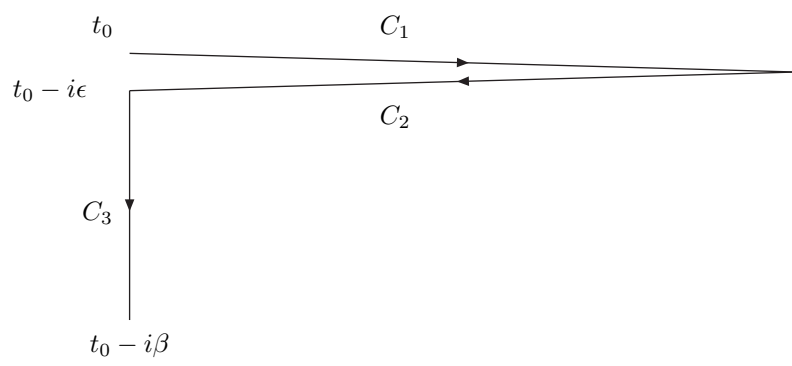


Fig. 1. Time path of real time thermal field theory